

MATRICES

* $\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \rightarrow$ diagonal matrix

* $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \rightarrow$ Scalar matrix
[$a=1 \rightarrow$ Unit matrix]

* $A+B=O \rightarrow$ Additive Inverse

* $A+O=A \rightarrow O$ is Additive Identity

* $A_{m \times n} B_{n \times p} \rightarrow C_{m \times p}$

* $AB \neq BA \rightarrow$ if $AB=BA \rightarrow$ commute matrices
if $AB=-BA \rightarrow$ anti-commute

* $|A|=0 \rightarrow A$ is singular

* $|A| \neq 0 \rightarrow$ Non singular

* $\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \quad a_{ij} = -a_{ji} \}$ skew symmetric
 $\downarrow$
 $A^T = -A$

Transpose

* $(A^T)^T = A$

* $(A+B)^T = A^T + B^T$

* $\lambda A^T = (\lambda A)^T$

* $(AB)^T = B^T A^T$

* $|A^T| = |A|$

* $A + A^T \rightarrow$ always symmetric

* $A - A^T \rightarrow$ always skew symm

* $A = \left[\frac{A+A^T}{2} \right] + \left[\frac{A-A^T}{2} \right]$

* If A_n is skew symm,

$n \rightarrow$ even $|A| = \lambda^2$

$n \rightarrow$ odd $|A| = 0$

Orthogonal

* $A \cdot A^T = A^T \cdot A = I$

* $|A| = \pm 1$

* If A is orthogonal, A^T must be orthogonal

* $\text{tr}(AA^T) =$ Sum of squares of all entries in A

* Idempotent $A^n = A, |A| = 0 \text{ (or) } 1, (A^2 = A)$

* Nilpotent $A^p = 0, |A| = 0$

$\rightarrow$ Smallest value of $p \rightarrow$ Nilpotent Index.

* Involutary $\rightarrow A^2 = I, |A| = \pm 1$

* Periodic $\rightarrow A^{k+1} = A \}$ least value of $k \rightarrow$ period

Adjoint Matrix

$\text{adj}(A) = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots \\ c_{21} & c_{22} & c_{23} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}^T = \begin{bmatrix} \text{Transpose of} \\ \text{co-factors} \\ \text{matrix} \end{bmatrix}$

* $A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = I |A|$

* $\text{adj}(A^T) = (\text{adj}(A))^T$

* $|\text{adj}(A)| = |A|^{n-1}$

* $AB = BA = I$

$B = A^{-1} (|A| \neq 0)$

* $A^{-1} = \frac{\text{adj}(A)}{|A|} [|A| \neq 0 \rightarrow \text{should be mentioned}]$

* $(A^{-1})^{-1} = A$ * $|A| = |A^T|$

* $(A^{-1})^T = (A^T)^{-1}$ * $\text{adj}(\text{adj} \dots r \text{ times } \text{adj}(KA))$

* $|A^{-1}| = \frac{1}{|A|} = K^{(n-1)r} \text{adj}(\text{adj} \dots r \text{ times } \text{adj}(A))$

* $\text{adj}(KA) = K^{n-1} \text{adj}(A)$

* $|\text{adj}(KA)| = K^{n(n-1)} |A|^{n-1}$

* $|\text{adj}(\text{adj}(\dots))| = |A|^{(n-1)^k}$

$k \text{ times}$

* $|\text{adj}(\text{adj}(KA))| = (K^n |A|)^{(n-1)^2}$

* $\text{adj}(\text{adj}(A)) = |A|^{n-2} A$

* A Symm $\rightarrow A^{-1}$ Symm

* A skew sym $\rightarrow A^{-1}$ skew sym (n is even only)

* $\text{adj}(AB) = \text{adj}(B) \cdot \text{adj}(A)$

* $(AB)^{-1} = B^{-1} \cdot A^{-1}$

$$* \text{adj}(\bar{A}^{-1}) = (\text{adj} A)^{-1} = \frac{A}{|A|}$$

NOTE:-

$$\triangleright \text{If } A = C \text{ then } BA = BC \checkmark$$

$$\Rightarrow AX = B \Rightarrow \bar{A}^{-1}AX = \bar{A}^{-1}B \text{ (provided } |A| \neq 0)$$

$\Rightarrow$ Applying det on B.S

$\Rightarrow$ Applying Inverse on B.S

$\Rightarrow$ Applying Adj on B.S

$\Rightarrow$ Applying Transpose on B.S

Characteristic Equation:-

$$* |A - \lambda I| \rightarrow \text{characteristic Eq} \\ \rightarrow \text{polynomial in terms of } \lambda \\ \text{with deg } n$$

* For a given $A_{n \times n}$, char Eq is

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_0 = 0$$

$$\Rightarrow \text{Sum of the roots} = \text{tr of } A$$

$$\Rightarrow \text{Product of roots} = |A|$$

$$\Rightarrow \text{Sum of roots taken 2 at a time} = \text{Sum of minors of diagonal} = \text{tr}(\text{adj} A)$$

System of Sol.

* Consistent $\rightarrow$ unique / infinite

* Trivial sol $\rightarrow (0, 0, 0)$

* Non-trivial $\rightarrow$ any sol other than $(0, 0, 0)$

* Trivial + Non trivial $\rightarrow$ Infinite

$$\text{Rule, } x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$$

Case-I $\Delta \neq 0 \rightarrow$ Unique

Case-II $\Delta = 0 \text{ \& } \Delta_1 = \Delta_2 = \Delta_3 = 0 \rightarrow$ Infinite sol

Case-III $\Delta = 0 \text{ \& } \Delta_1, \Delta_2, \Delta_3 \text{ at least one of them is non-zero} \rightarrow$ No sol

* Before using, check for parallel

* If $A = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ then

$$\bar{A}^{-1} = \text{diag}(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}) \quad (\lambda_i \neq 0)$$

$$* A \cdot \bar{A}^T = 0 \Rightarrow A = 0$$

$$* |AB| = |B||A|$$

$$* \text{adj}(AB) = \text{adj}(B) \times \text{adj}(A)$$

* If $AB = 0$, it doesn't imply that either $A = 0$ / $B = 0$
But if given $|B| \neq 0$ then A must be null matrix.

* If $AB = 0$ then $BA = 0$

If $AB = I$ then $BA = I$

* For every column matrix X , if $X^T A X = 0$
then A must be Skew-symmetric $A = -A^T$

$$* \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \text{Inverse} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

* $(\bar{A})^T \rightarrow A^0$ (conjugate matrix)

If $A = A^0 \rightarrow$ hermitian matrix

If $A^0 = -A \rightarrow$ skew hermitian matrix

$$* (AB)^0 = B^0 \cdot A^0$$

Trace

$$*) \text{Tr}(A+B) = \text{Tr} A + \text{Tr}(B)$$

$$*) \text{Tr}(AB) = \text{Tr}(BA)$$

$$*) \text{Tr}(A^T) = \text{Tr}(A)$$

DETERMINANTS

* $\Delta = \sum a_{ij} \cdot c_{ij}$

* If $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \Delta \Rightarrow \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = \Delta^2 \text{ (3x3)}$

* Cofactors
det = Δ^{n-1}

* $|k[A]| = k^n |A|$

Standard results

1) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc) = -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

2) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

3) $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

4) $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

5) $\begin{vmatrix} 1 & a & a^4 \\ 1 & b & b^4 \\ 1 & c & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)(a^2 + b^2 + c^2 + ab + bc + ca)$

6) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

7) $\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ac \\ 1 & c^2 & ab \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

8) $\begin{vmatrix} a^2+1 & ba & ac \\ ab & 1+b^2 & bc \\ ac & bc & 1+c^2 \end{vmatrix} = 1 + a^2 + b^2 + c^2$

* $a_{11}c_{31} + a_{12}c_{32} + a_{13}c_{33} = 0$